17 Fourier Series: Magnitude & Phase Spectrum

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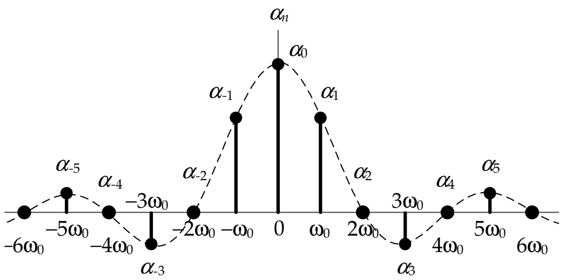
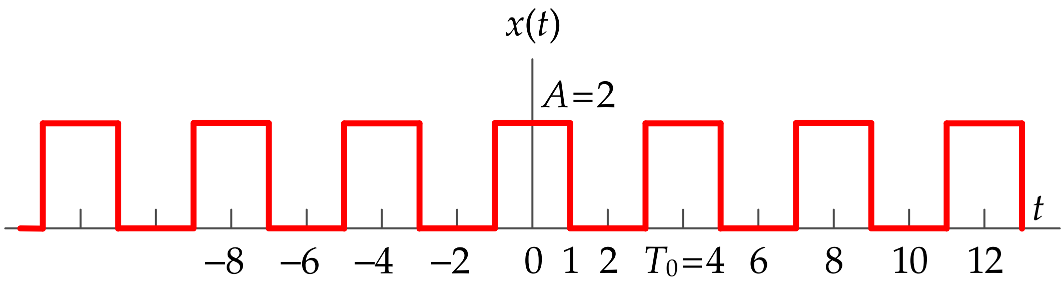


Remember earlier example periodic signal $x(t) = \text{rep}_4\left\{2\ \text{rect}\left(\frac{t}{2}\right)\right\}$



For which the complex exponential Fourier series coefficients were

$$\alpha_n = \frac{A\tau}{T_0}\text{sinc}\left(\frac{n\omega_0\tau}{2\pi}\right) = \text{sinc}\left(\frac{n}{2}\right)$$

$$\alpha_0 = \text{sinc}\left(\frac{0}{2}\right) = 1 \qquad |\alpha_0| = 1 \qquad \angle\alpha_0 = 0$$

$$\alpha_1 = \text{sinc}\left(\frac{1}{2}\right) = \frac{\sin(\pi/2)}{\pi/2} = \frac{+1}{\pi/2} \qquad |\alpha_1| = \frac{2}{\pi} = 0.6366 \qquad \angle\alpha_1 = 0$$

$$\alpha_2 = \text{sinc}\left(\frac{2}{2}\right) = \frac{\sin(\pi)}{\pi} = \frac{0}{\pi} \qquad |\alpha_2| = 0 \qquad \angle\alpha_2 = 0$$

$$\alpha_3 = \text{sinc}\left(\frac{3}{2}\right) = \frac{\sin(3\pi/2)}{3\pi/2} = \frac{-1}{3\pi/2} \qquad |\alpha_3| = \frac{2}{3\pi} = 0.2122 \qquad \angle\alpha_3 = \pm\pi$$

$$\alpha_4 = \text{sinc}\left(\frac{4}{2}\right) = \frac{\sin(2\pi)}{2\pi} = \frac{0}{2\pi} \qquad |\alpha_4| = 0 \qquad \angle\alpha_4 = 0$$

$$\alpha_5 = \text{sinc}\left(\frac{5}{2}\right) = \frac{\sin(5\pi/2)}{5\pi/2} = \frac{+1}{5\pi/2} \qquad |\alpha_5| = \frac{2}{5\pi} = 0.1273 \qquad \angle\alpha_5 = 0$$

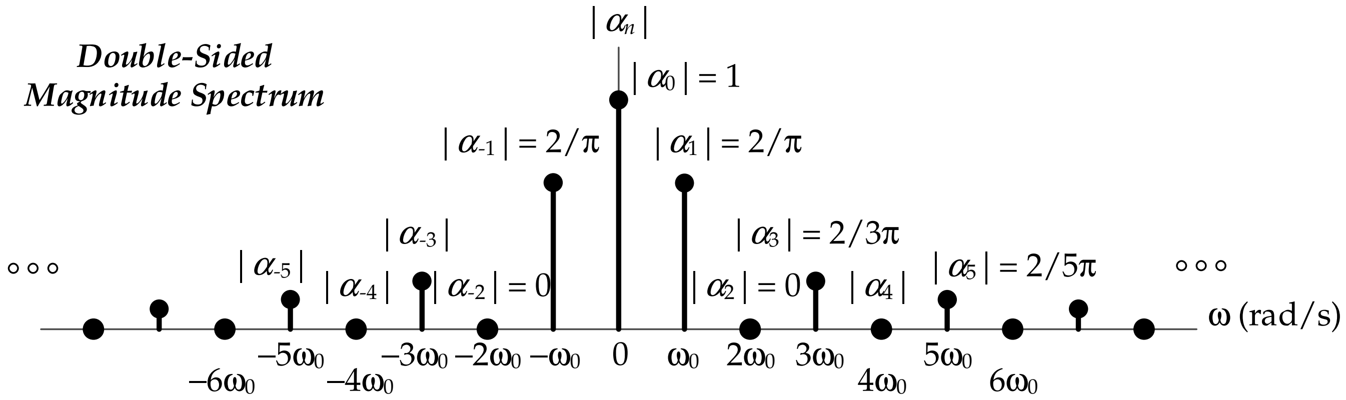
….

To visualize the harmonics that are scaled by the coefficients α_n, i.e.,

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, e^{jn\omega_0 t} \,, \qquad \omega_0 = \frac{2\pi}{T_0}$$

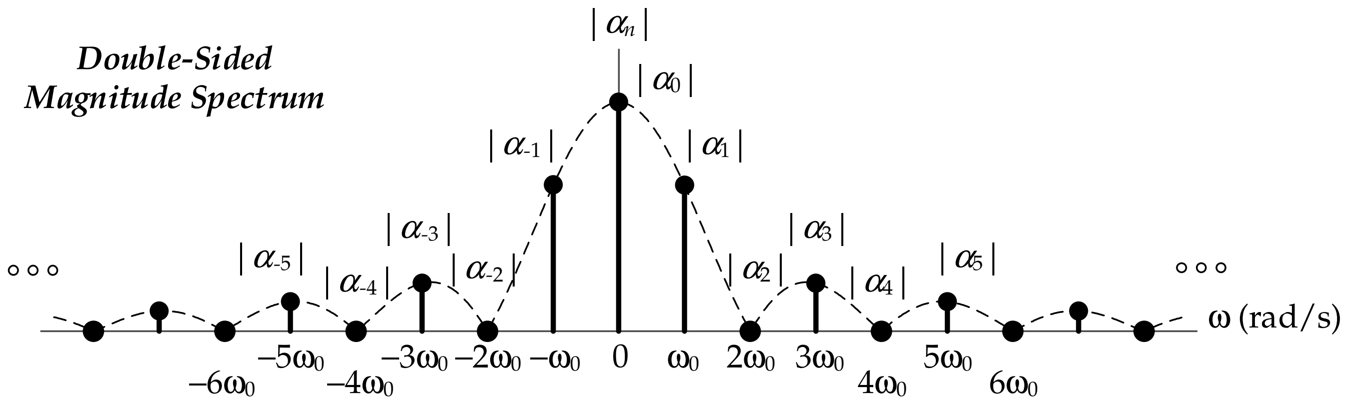
we typically plot the magnitude $|\alpha_n|$ of the complex number α_n for the nth harmonic versus the frequency of such harmonic, which is $\omega = n\omega_0$ (in a sense, similar to plotting it versus n).

This is called the **magnitude spectrum** of $x(t)$. It is also known as **double-sided magnitude spectrum**, **two-sided magnitude spectrum**, and sometimes amplitude spectrum or double-sided amplitude spectrum or two-sided amplitude spectrum or line spectrum.



Notice that if $x(t)$ is real-valued signal, then $\alpha_{-n} = \alpha_n^*$, which means that $|\alpha_{-n}| = |\alpha_n|$ and $\angle\alpha_{-n} = -\angle\alpha_n$, and hence the **magnitude spectrum** has **even symmetry**.

The envelope is not part of α_n, but it helps visualize the behavior of α_n. It also helps calculate the bandwidth W of the signal $x(t)$ [see later].



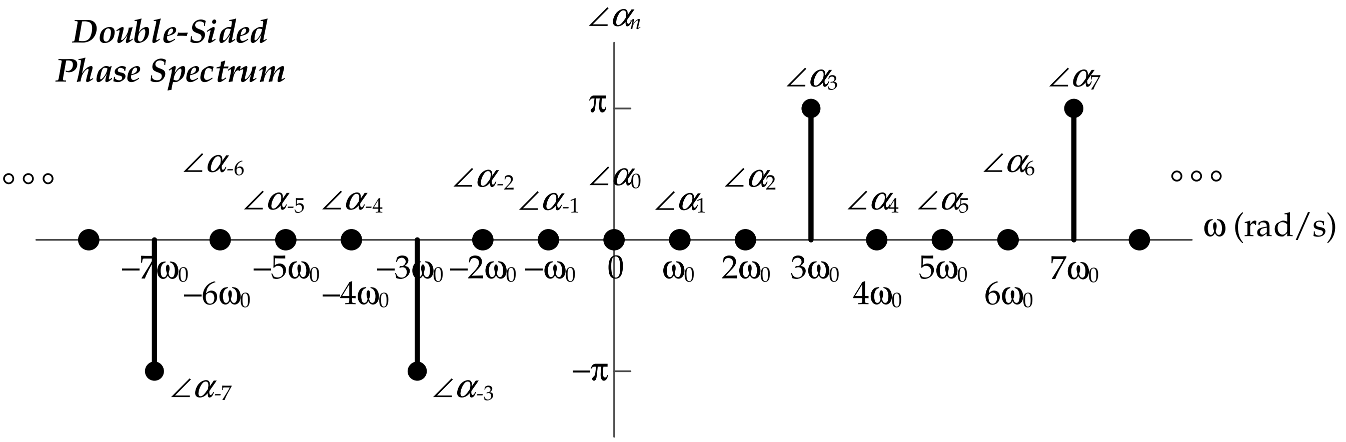
The separation (spacing) between the lines is the fundamental frequency $\omega_0 = 2\pi/T_0$ of the periodic signal $x(t)$.

We also plot the phase $\measuredangle\alpha_n$ of the complex coefficient for the nth harmonic versus the harmonic frequency $n\omega_0$.

This is called the **phase spectrum** of $x(t)$ or **double-sided phase spectrum**, or **two-sided phase spectrum**.

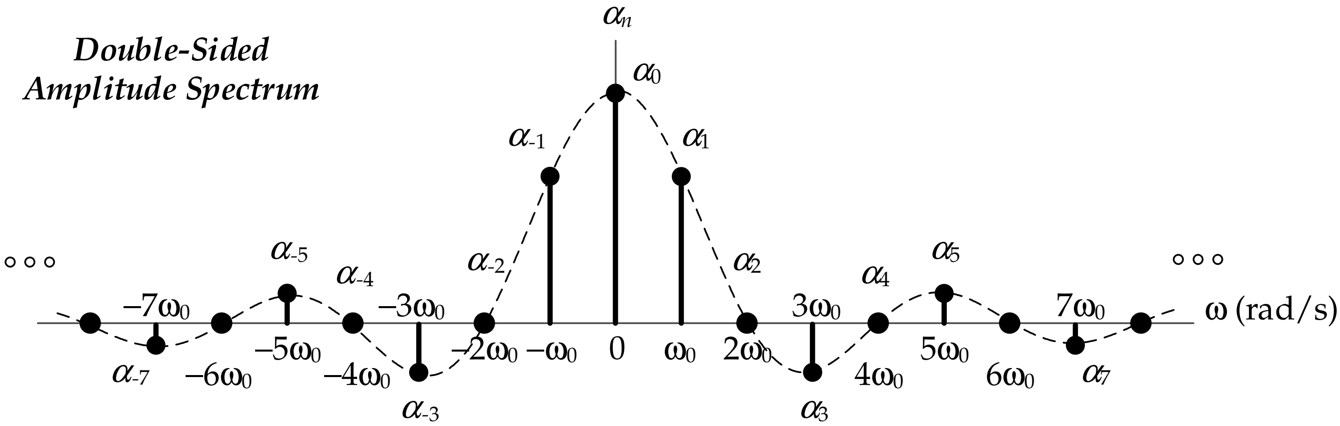
Notice that if $x(t)$ is real-valued signal, then $\alpha_{-n} = \alpha_n^*$, which means that $|\alpha_{-n}| = |\alpha_n|$ and $\angle\alpha_{-n} = -\angle\alpha_n$, and hence the **phase spectrum** has **odd symmetry**.

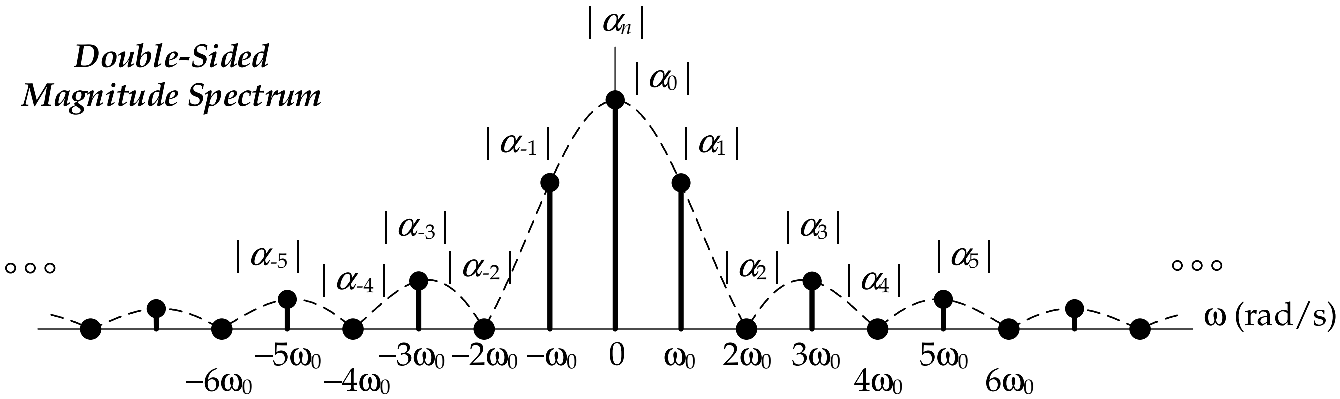
The two plots together (magnitude spectrum and phase spectrum) are the **frequency spectra** of $x(t)$, also known as its frequency-domain representation.



The separation (spacing) between the lines is the fundamental frequency ω_0 of the periodic signal $x(t)$.

When α_n are real positive/negative numbers (i.e., complex numbers with phase shift of 0 or $\pm\pi$), the magnitude spectrum and phase spectrum are, sometimes, combined into one diagram, called amplitude spectrum.





Line at zero frequency (α_0) is the DC value of $x(t)$ or its average value.

Every pair of lines at frequency $\omega = n\omega_0$ and $\omega = -n\omega_0$ represent one cosine signal (the nth harmonic). To get a cosine we need a pair of lines.

No such thing as negative frequency, rather since $\alpha_{-n} = \alpha_n^*$, we have

$$\alpha_n \, e^{jn\omega_0 t} + \alpha_{-n} \, e^{-jn\omega_0 t}$$

$$= |\alpha_n| e^{j\measuredangle\alpha_n} e^{jn\omega_0 t} + |\alpha_{-n}| e^{j\measuredangle\alpha_{-n}} \, e^{-jn\omega_0 t}$$

$$= |\alpha_n| e^{j\measuredangle\alpha_n} e^{jn\omega_0 t} + |\alpha_n| e^{-j\measuredangle\alpha_n} \, e^{-jn\omega_0 t}$$

$$= |\alpha_n| e^{j(n\omega_0 t + \measuredangle\alpha_n)} + |\alpha_n| \, e^{-j(n\omega_0 t + \measuredangle\alpha_n)}$$

$$= 2|\alpha_n| \left[\frac{e^{j(n\omega_0 t + \measuredangle\alpha_n)} + e^{-j(n\omega_0 t + \measuredangle\alpha_n)}}{2} \right]$$

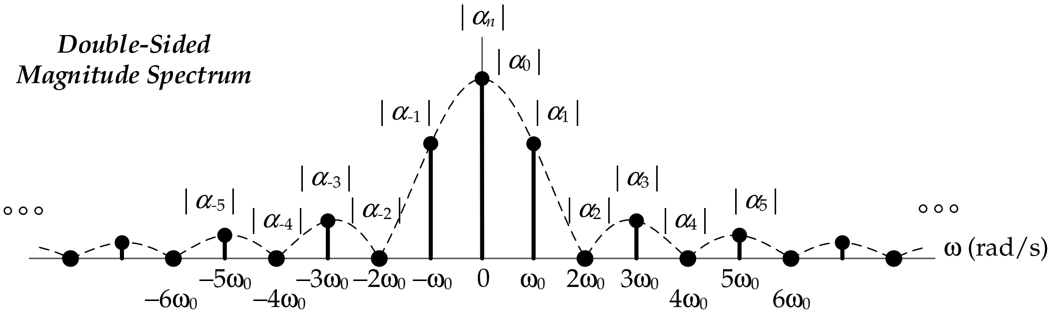
$$= c_n \cos(n\omega_0 t - \theta_n)$$

The magnitude & phase spectra are saying that the periodic signal $x(t)$ can be expressed as sum of DC (zero freq.) plus sinusoids (harmonics) of frequencies: ω_0, $2\omega_0$, $3\omega_0$, . . ., $n\omega_0$, . . . with varying amplitudes and phase shifts for the different harmonics (as shown in the spectra).

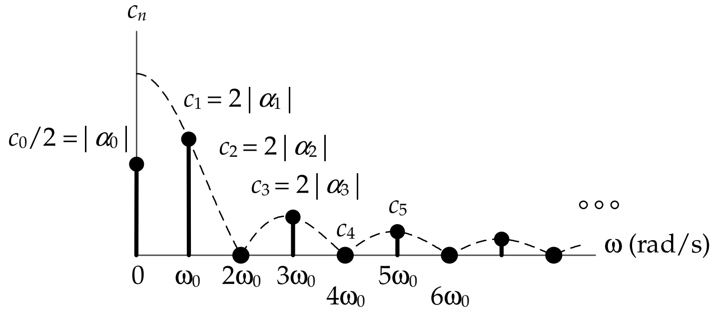
If we plot the amplitudes of the compact Fourier series coefficients c_n, for the nth harmonic versus the frequency of such harmonic $\omega = n\omega_0$ we get the **single-sided magnitude spectrum**, or **one-sided magnitude spectrum**, or sometimes single-sided amplitude spectrum or one-sided amplitude spectrum.

If we plot the phase shift of the compact Fourier series coefficients θ_n, for the nth harmonic versus the frequency of such harmonic $\omega = n\omega_0$ we get the **single-sided phase spectrum** or **one-sided phase spectrum**.

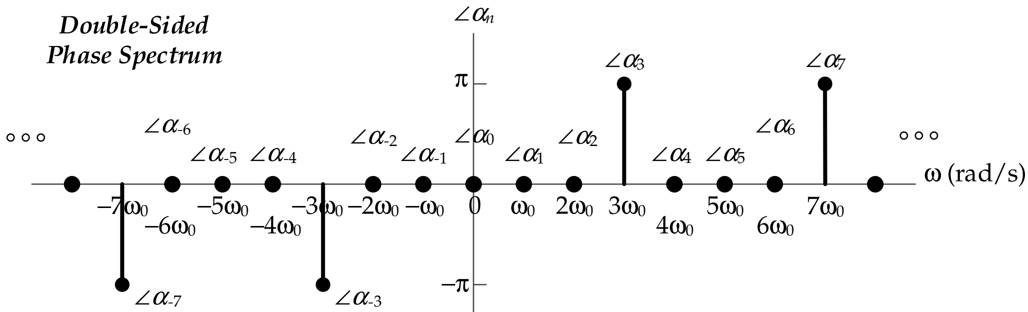
Double-Sided Magnitude Spectrum



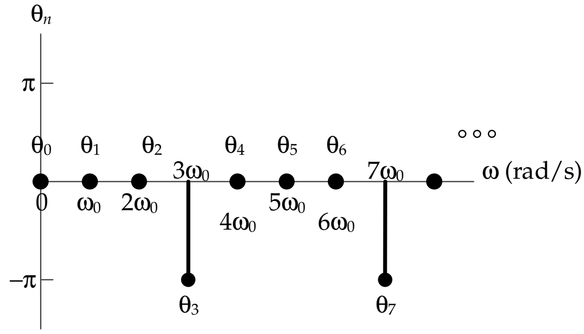
Single-Sided Magnitude Spectrum



Double-Sided Phase Spectrum



Single-Sided Phase Spectrum



Complex exponential Fourier series

$$x(t) = \sum_{n=-\infty}^{\infty} \alpha_n \, e^{jn\omega_0 t} \, , \qquad \omega_0 = \frac{2\pi}{T_0}$$

Trigonometric Fourier series

$$x(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t)] \, , \quad \omega_0 = \frac{2\pi}{T_0}$$

Compact Fourier series

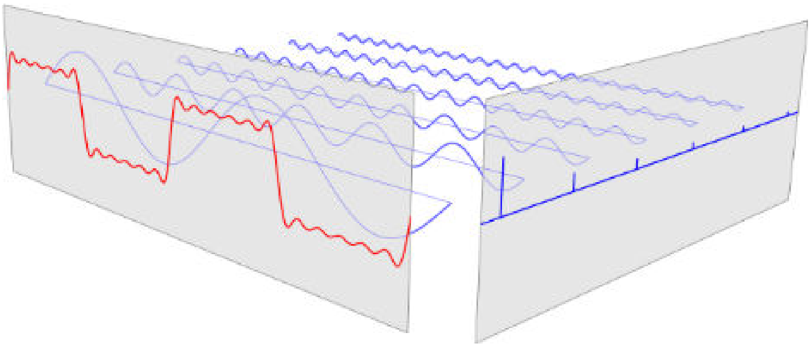
$$x(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega_0 t - \theta_n) \, , \qquad \omega_0 = \frac{2\pi}{T_0}$$

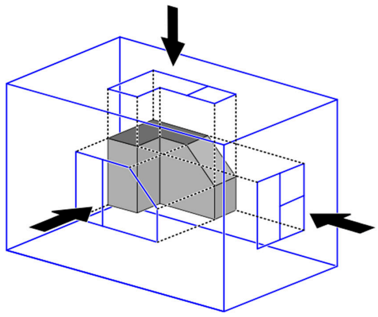
We say the frequency spectra of $x(t)$ show the **frequency content** of the signal $x(t)$. They represent an alternative (but equivalent) way of describing the periodic signal $x(t)$ instead of as a function of t, rather via its harmonic content (constituent frequencies).

The frequency spectra of a signal are called the **frequency-domain** representation of $x(t)$, in contrast to the time-domain representation, where $x(t)$ is specified as a function of time t.

These spectra represent a **unique** description of the specific periodic signal $x(t)$, hence knowing the magnitude and phase spectra we can reconstruct the signal $x(t)$ [using the Fourier series formula].

Visualizing this frequency content can be very helpful.





Bandwidth of signal $x(t)$: range (or band) of frequencies (must be positive frequencies) that contain the significant harmonics within the signal $x(t)$ [i.e., those with large enough magnitude or average power].

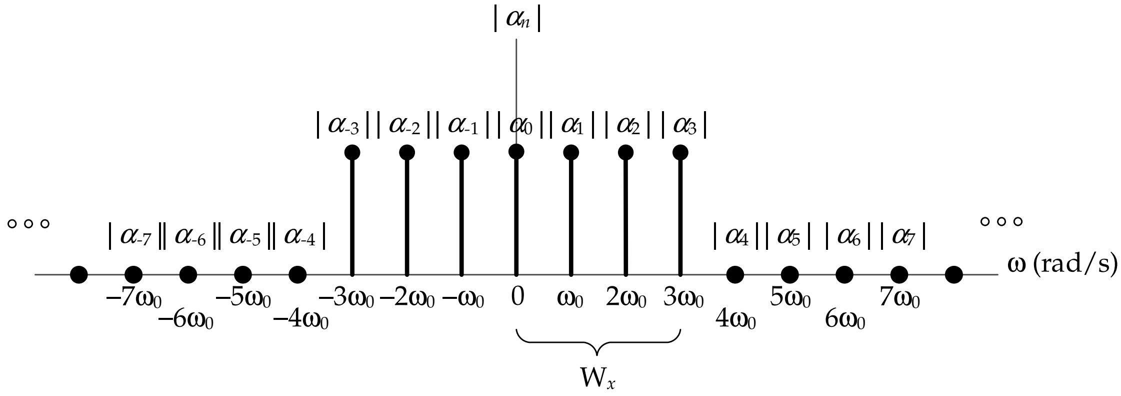
We denote bandwidth of $x(t)$ by B_x (when measured in Hz) or $W_x = 2\pi B_x$ (when measured in rad/s).

Be careful when calculating bandpass bandwidth versus baseband bandwidth.

Be careful to distinguish between the bandwidth of a filter (bandwidth of a device) versus the bandwidth of the signal passing through the filter.

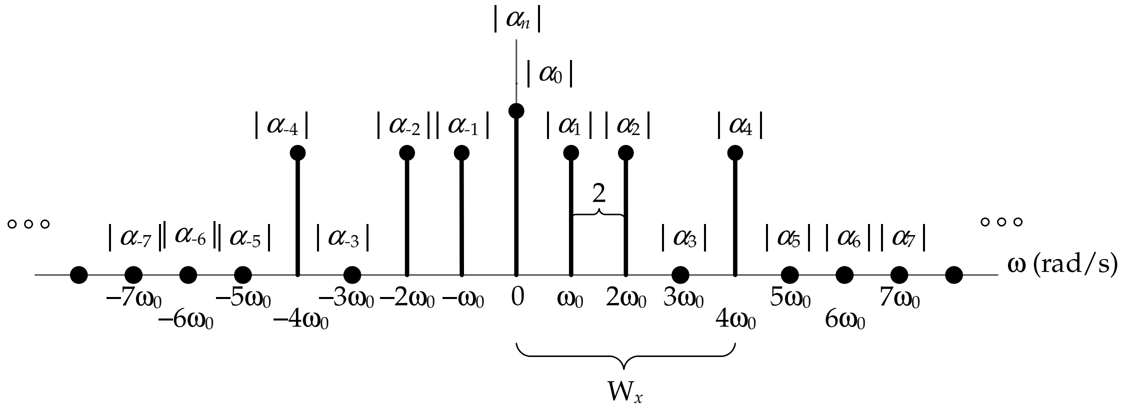
Knowing the bandwidth of the signal can be very important. For example, in sampling, filtering, communication systems, etc.

Q1. For the following frequency-domain representation of $x(t)$, where $\alpha_n = \left\{5 \text{ rect}\left(\frac{n\omega_0}{6\pi}\right)\right\}$, determine the fundamental frequency of the signal and its bandwidth.



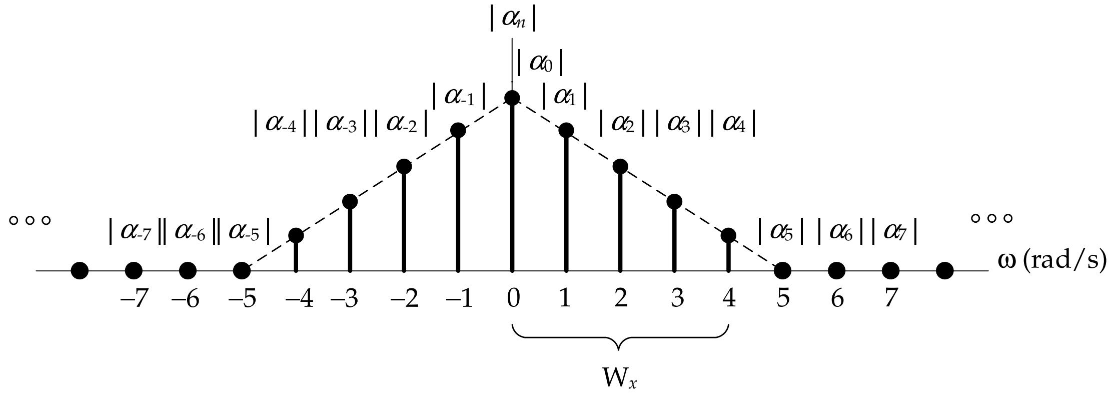
Answer. $\omega_0 = \pi$ [rad/s], $W_x = 3\omega_0 = 3\pi$ [rad/s].

Q2. For the following frequency-domain representation of $x(t)$, determine the fundamental frequency of the signal and its bandwidth.



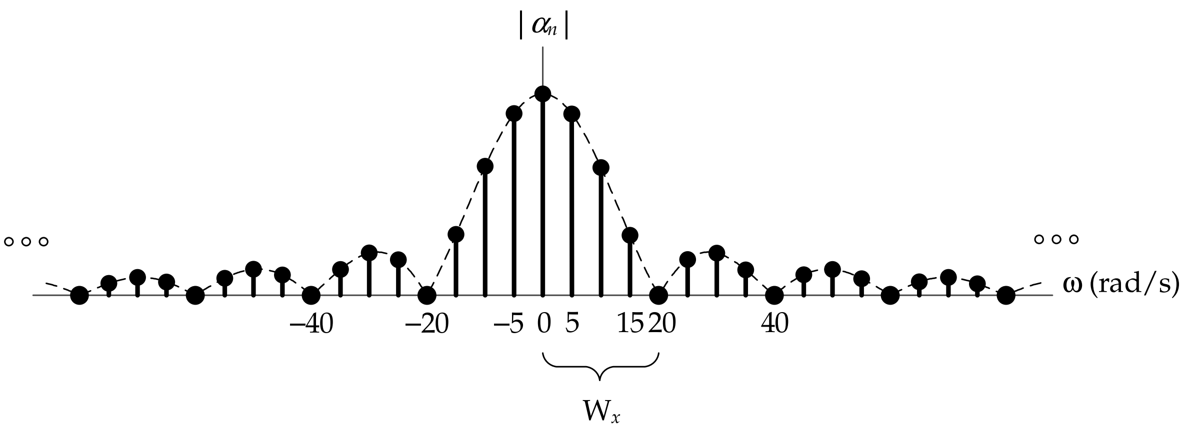
Answer. $\omega_0 = 2$ [rad/s], $W_x = 4\omega_0 = 8$ [rad/s].

Q3. For the following frequency-domain representation of $x(t)$, where $\alpha_n = \left\{7\,\Delta\left(\frac{n\omega_0}{5}\right)\right\}$, determine the fundamental frequency of the signal and its bandwidth.



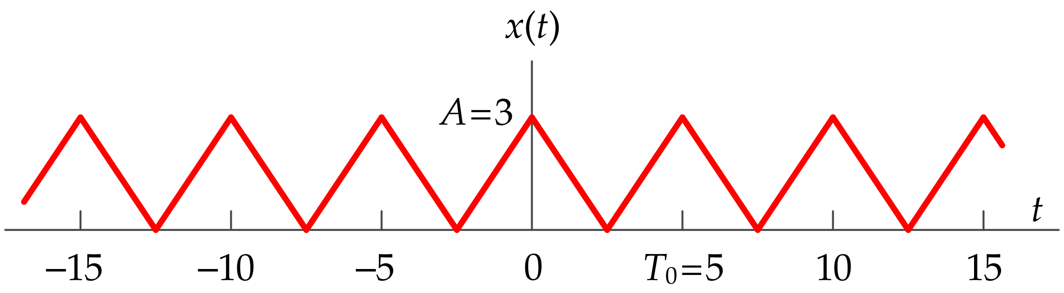
Answer. $\omega_0 = 1$ [rad/s], $W_x = 4\omega_0 = 4$ [rad/s].

Q4. For the following frequency-domain representation of $x(t)$, determine the fundamental frequency of the signal and its bandwidth.



Answer. $\omega_0 = 5$ [rad/s], $W_x = 4\omega_0 = 20$ [rad/s].

Q5. For the following signal $x(t) = \text{rep}_5\left\{3\,\Delta\left(\frac{t}{2.5}\right)\right\}$, draw the double-sided magnitude spectrum and double-sided phase spectrum, then determine the bandwidth of the signal.



Q5. Solution. Need the complex exponential Fourier series coefficients for double-sided spectra

$$\alpha_n = \frac{A\tau}{T_0}\text{sinc}^2\left(\frac{n\omega_0\tau}{2\pi}\right) = \frac{3}{2}\text{sinc}^2\left(\frac{n}{2}\right)$$

Bandwidth is at the first null of the $\text{sinc}^2\left(\frac{n\omega_0\tau}{2\pi}\right)$

$$W_x = n\omega_0 = \frac{2\pi}{\tau} = \frac{2\pi \times 2}{T_0} = \frac{2\pi}{2.5} = 2.5133\ [\text{rad/s}]$$

$$B_x = \frac{W_x}{2\pi} = \frac{1}{\tau} = \frac{2}{T_0} = \frac{1}{2.5} = 0.4\ [\text{Hz}]$$

$$\alpha_0 = \frac{3}{2}\text{sinc}^2\left(\frac{0}{2}\right) = \frac{3}{2} \qquad |\alpha_0| = 1.5 \qquad \angle\alpha_0 = 0$$

$$\alpha_1 = \frac{3}{2}\text{sinc}^2\left(\frac{1}{2}\right) = \frac{3}{2}\frac{\sin^2(\pi/2)}{(\pi/2)^2} = \frac{6}{\pi^2} \qquad |\alpha_1| = 0.6079 \qquad \angle\alpha_1 = 0$$

$$\alpha_2 = \frac{3}{2}\text{sinc}^2\left(\frac{2}{2}\right) = \frac{3}{2}\frac{\sin^2(\pi)}{(\pi)^2} = \frac{3\times 0}{2\pi^2} \qquad |\alpha_2| = 0 \qquad \angle\alpha_2 = 0$$

$$\alpha_3 = \frac{3}{2}\text{sinc}^2\left(\frac{3}{2}\right) = \frac{3}{2}\frac{\sin^2(3\pi/2)}{(3\pi/2)^2} = \frac{2}{3\pi^2} \qquad |\alpha_3| = 0.0675 \qquad \angle\alpha_3 = 0$$

$$\alpha_4 = \frac{3}{2}\text{sinc}^2\left(\frac{4}{2}\right) = \frac{3}{2}\frac{\sin^2(2\pi)}{(2\pi)^2} = \frac{3\times 0}{8\pi^2} \qquad |\alpha_4| = 0 \qquad \angle\alpha_4 = 0$$

$$\alpha_5 = \frac{3}{2}\text{sinc}^2\left(\frac{5}{2}\right) = \frac{3}{2}\frac{\sin^2(5\pi/2)}{(5\pi/2)^2} = \frac{6}{25\pi^2} \qquad |\alpha_5| = 0.0243 \qquad \angle\alpha_5 = 0$$

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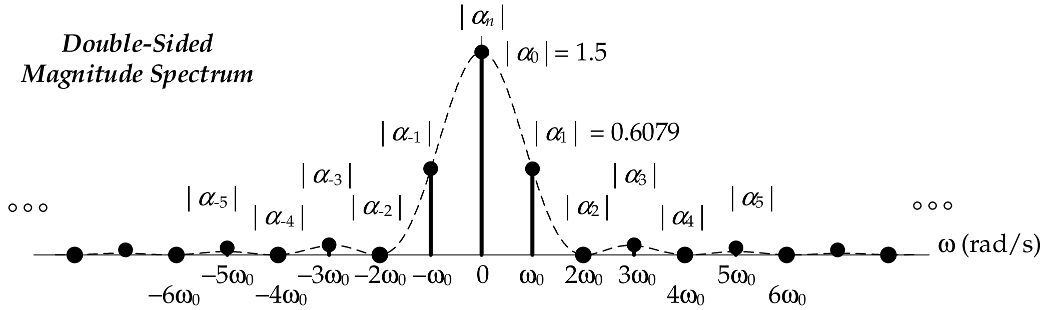
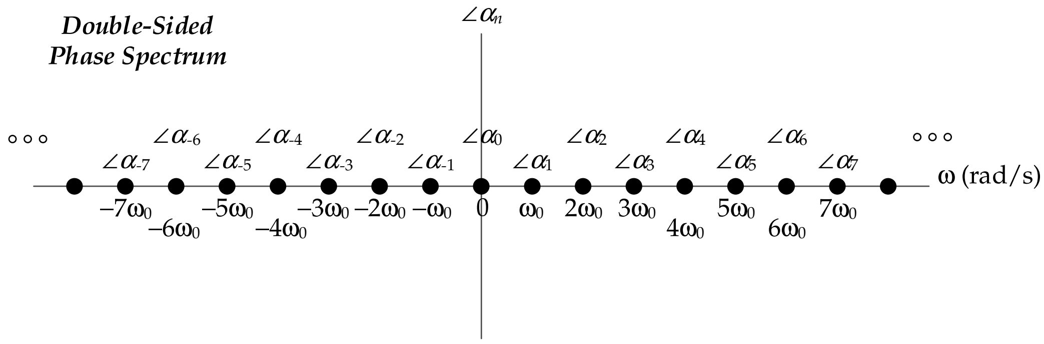
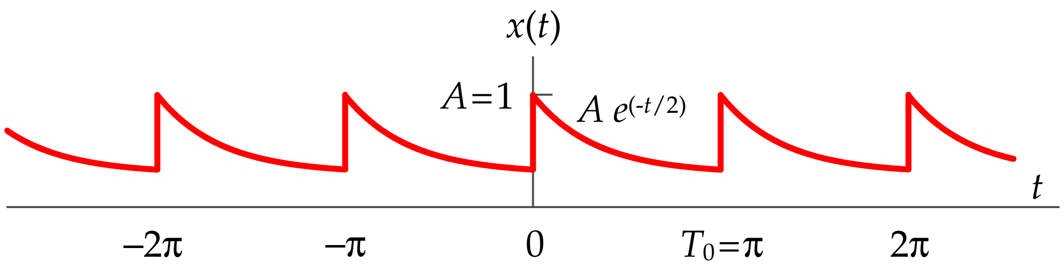




Q6. For the following signal $x(t) = \text{rep}_{T_0}\left\{A\, e^{(-t/2)}\, \text{rect}\left(\frac{t-T_0/2}{T_0}\right)\right\}$, draw the single-sided magnitude spectrum and single-sided phase spectrum.



Q6. Solution. Need compact Fourier series coefficients for single-sided spectra

$$\alpha_0 = 0.5043$$

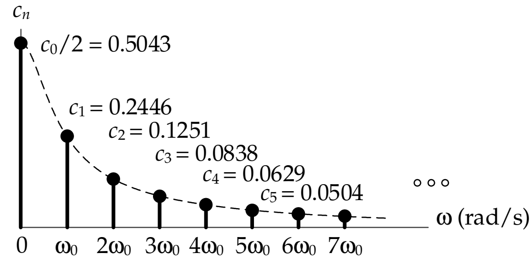
$$\alpha_n = \frac{a_n}{2} - j\frac{b_n}{2} = 0.5043\left(\frac{1}{1+16n^2}\right) - j0.5043\left(\frac{4n}{1+16n^2}\right), \quad n \neq 0$$

$$\frac{c_0}{2} = 0.5043$$

$$c_n = 0.5043\left(\frac{2}{\sqrt{1+16n^2}}\right)$$

$$\theta_n = \tan^{-1}(4n) \quad \text{[fourth quadrant]}$$

Single-Sided
Magnitude Spectrum



Single-Sided
Phase Spectrum